Electronic Journal of Qualitative Theory of Differential Equations
2016, No. 37, 1–12; doi: 10.14232/ejqtde.2016.1.37 http://www.math.u-szeged.hu/ejqtde/

Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential

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Received 10 April 2016, appeared 6 June 2016
Communicated by Patrizia Pucci

Abstract. We are concerned with the existence of entire distributional nontrivial solutions for a new class of nonlinear partial differential equations. The differential operator was introduced by A. Azzolini *et al.* [3,4] and it is described by a potential with different growth near zero and at infinity. The main result generalizes a property established by P. Rabinowitz in relationship with the existence of nontrivial standing waves of the Schrödinger equation with lack of compactness. The proof combines arguments based on the mountain pass and energy estimates.

Keywords: nonlinear Schrödinger equation, nonhomogeneous differential operator, mountain pass.

2010 Mathematics Subject Classification: Primary: 35J60. Secondary: 58E05.

1 Introduction

The Schrödinger equation has a basic role in quantum theory and it plays the role of Newton's conservation laws of energy in classical mechanics, that is, it predicts the future behaviour of a dynamical system. The *linear* Schrödinger equation provides a thorough description of particles in a non-relativistic setting. The structure of the *nonlinear* form of the Schrödinger equation is much more complicated. The most common applications of this equation vary from Bose–Einstein condensates and nonlinear optics, stability of Stokes waves in water, propagation of the electric field in optical fibers to the self-focusing and collapse of Langmuir waves in plasma physics and the behaviour of deep water waves and freak waves (or rogue waves) in the ocean. The nonlinear Schrödinger equation also describes phenomena arising in the theory of Heisenberg ferromagnets and magnons, self-channelling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields, plasma physics (e.g., the Kurihara superfluid film equation). We refer to M. J. Ablowitz,

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B. Prinari and A. D. Trubatch [1], T. Cazenave [10], C. Sulem and P. L. Sulem [26] for a modern overview and relevant applications.

In a seminal paper, P. Rabinowitz [22] showed how variational arguments based on the mountain pass theorem can be applied to obtain existence results for nonlinear Schrödinger-type equations with lack of compactness. P. Rabinowitz [22] studied the nonlinear Schrödinger equation

$$-\Delta u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N \ (N \geq 3), \tag{1.1}$$

where a is a positive potential and f has a subcritical growth. The existence of nontrivial standing waves of problem (1.1) strongly relies on the mountain pass theorem.

We point out that the mountain pass theorem was established by A. Ambrosetti and P. Rabinowitz [2] and it is a basic tool in nonlinear analysis. The limiting version of this result, which corresponds to *mountains of zero altitude* is due to P. Pucci and J. Serrin [17–19]. We also refer to H. Brezis and L. Nirenberg [9] who proved a version of the mountain pass theorem that includes the limiting case corresponding to mountains of zero altitude. Their proof combines a pseudo-gradient lemma, an original perturbation argument and the Ekeland variational principle [11]. For further related results and applications of the mountain pass theorem, we refer to Y. Jabri [13], I. Peral [15], P. Pucci and V. Rădulescu [16], D. Repovš [24], and M. Roșiu [25].

Problems like (1.1) are obtained by substituting the ansatz

$$\psi(x,t) = \exp(-iEt/\hbar)u(x)$$

into the Schrödinger equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where $1 < p < (N+2)/(N-2)$. Here, $\hbar$ is the Plank constant divided by 2π, ψ is the wave function, m is the magnetic quantum number, V is the potential energy, and γ is a constant that depends on the number of particles.

Under natural hypotheses, P. Rabinowitz [22] proved that problem (1.1) has a nontrivial distributional solution. This result was generalized by F. Gazzola and V. Rădulescu [12] in two nonsmooth settings (Degiovanni and Clarke theories) and by M. Mihăilescu and V. Rădulescu [14] in the framework of singular potentials of Hardy or Caffarelli–Kohn–Nirenberg type.

In some recent papers, A. Azzollini *et al.* [3, 4] introduced a new class of differential operators with a variational structure. They considered nonhomogeneous operators of the type

$$\operatorname{div}[\phi'(|\nabla u|^2)\nabla u],$$

where $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ has a different growth near zero and at infinity. Such a behaviour occurs if $\phi(t) = 2[\sqrt{1+t} - 1]$, which corresponds to the prescribed mean curvature operator (capillary surface operator), which is defined by

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

More generally, $\phi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, where $1 < p < q < N$. Such a behaviour is fulfilled if

$$\phi(t) = \frac{2}{p}\left[(1+t^{q/2})^{p/q} - 1\right],$$

which generates the differential operator

$$\operatorname{div}\left[(1+|\nabla u|^q)^{(p-q)/q}|\nabla u|^{q-2}\nabla u\right].$$

The main purpose of this paper is to study problem (1.1) in the new abstract setting introduced by A. Azzollini *et al.* [3,4]. In the next section, we introduce the main hypotheses and we state the basic result of this paper. The proof and related comments are developed in the final section of this paper.

2 The main result

We study the following quasilinear Schrödinger equation

$$-\operatorname{div}[\phi'(|\nabla u|^2)\nabla u]+a(x)|u|^{\alpha-2}u=f(x,u) \quad \text{in } \mathbb{R}^N \ (N\geq 3). \tag{2.1}$$

Throughout this paper we assume that α, p, q, and s are real numbers satisfying the following properties:

$$\begin{cases} 1<p<q<N \\ 1<\alpha<\dfrac{p^*q'}{p'} \\ \max\{\alpha,q\}<s<p^*:=\dfrac{pN}{N-p}, \end{cases} \tag{2.2}$$

where p' denotes the conjugate exponent of p, that is, $p'=p/(p-1)$.

We assume that the potential a in (2.1) is singular and that it satisfies the following hypotheses:

(a_1) $a\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ and $\operatorname{ess\,inf}_{\mathbb{R}^N} a>0$;

(a_2) $\lim_{x\to 0}a(x)=\lim_{|x|\to\infty}a(x)=+\infty$.

A potential satisfying these conditions is $a(x)=\exp(|x|)/|x|$, for $x\in\mathbb{R}^N\setminus\{0\}$.

We assume that the nonlinearity $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function characterized by the following conditions:

(f_1) $f(x,u)=o(u^{\alpha-1})$ as $u\to 0^+$, uniformly for a.e. $x\in\mathbb{R}^N$;

(f_2) $f(x,u)=O(u^{s-1})$ as $u\to\infty$, uniformly for a.e. $x\in\mathbb{R}^N$;

(f_3) there exists $\theta>\alpha$ such that $0<\theta F(x,u)\leq uf(x,u)$ for all $u>0$, a.e. $x\in\mathbb{R}^N$, where $F(x,u)=\int_0^u f(x,t)dt$;

(f_4) if $\alpha<q$ then $\lim_{u\to+\infty}F(x,u)/u^q=+\infty$ uniformly for a.e. $x\in\mathbb{R}^N$.

Next, we assume that the differential operator in problem (2.1) is generated by the function $\phi\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ having the following properties:

(ϕ_1) $\phi(0)=0$;

(ϕ_2) there exists $c_1>0$ such that $\phi(t)\geq c_1t^{p/2}$ if $t\geq 1$ and $\phi(t)\geq c_1t^{q/2}$ if $0\leq t\leq 1$;

(ϕ_3) there exists $c_2>0$ such that $\phi(t)\leq c_2t^{p/2}$ if $t\geq 1$ and $\phi(t)\leq c_2t^{q/2}$ if $0\leq t\leq 1$;

(ϕ_4) there exists $0 < \mu < 1/s$ such that $2t\phi'(t) \le s\mu\phi(t)$ for all $t \ge 0$;

(ϕ_5) the mapping $t \mapsto \phi(t^2)$ is strictly convex.

Since our hypotheses allow that ϕ' approaches 0, problem (2.1) is *degenerate* and no ellipticity condition is assumed.

In order to state the main abstract result of this paper, we need to describe the functional setting corresponding to problem (2.1).

In what follows, we denote by $\|\cdot\|_r$ the Lebesgue norm for all $1 \le r \le \infty$ and by $C_c^\infty(\mathbb{R}^N)$ the space of all C^∞ functions with a compact support.

Definition 2.1. We define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{L^p+L^q} := \inf\{\|v\|_p + \|w\|_q;\ v \in L^p(\Omega),\ w \in L^q(\Omega),\ u = v + w\}.$$

For more properties of the Orlicz space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ we refer to M. Badiale, L. Pisani, and S. Rolando [6, Section 2].

Throughout this paper we denote

$$\|u\|_{p,q} = \|u\|_{L^p+L^q}.$$

A key role in our arguments is played by the function space

$$\mathcal{X} := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| := \|\nabla u\|_{p,q} + \left(\int_{\mathbb{R}^N} a(x)|u|^\alpha dx\right)^{1/\alpha}.$$

We notice that $\mathcal{X}$ is continuously embedded in the reflexive Banach space $\mathcal{W}$ defined in [4, p. 202], where $\mathcal{W}$ is the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm $\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha$.

Definition 2.2. A weak solution of problem (2.1) is a function $u \in \mathcal{X} \setminus \{0\}$ such that for all $v \in \mathcal{X}$

$$\int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)\nabla u \nabla v + a(x)|u|^{\alpha-2}uv - f(x,u)v\right] dx = 0.$$

The energy functional associated to problem (2.1) is $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ defined by

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Our assumptions imply that

$$\phi(|\nabla u|^2) \simeq \begin{cases} |\nabla u|^p, & \text{if } |\nabla u| \gg 1; \\ |\nabla u|^q, & \text{if } |\nabla u| \ll 1. \end{cases}$$

We also observe that $\mathcal{E}$ is well-defined on $\mathcal{X}$, of class C^1 (see also A. Azzollini [3, Theorem 2.5]). Moreover, for all $u, v \in \mathcal{X}$ its Gâteaux directional derivative is given by

$$\mathcal{E}'(u)(v) = \int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)\nabla u \nabla v + a(x)|u|^{\alpha-2}uv - f(x,u)v\right] dx.$$

The main result of this paper establishes the following existence property.

Theorem 2.3. *Assume that hypotheses (2.2), (a_1), (a_2), (f_1)–(f_4), and (ϕ_1)–(ϕ_5) are fulfilled. Then problem (2.1) admits at least one weak solution.*

We point out that a related existence property was established by A. Azzollini, P. d'Avenia, and A. Pomponio [4, Theorem 1.3] but under the assumption that the potential a reduces to a positive constant. Our setting is different and it corresponds to variable potentials which blow-up both at the origin and at infinity. The lack of compactness due to the unboundedness of the domain is handled in [4] by restricting the study to the case of *radially symmetric* weak solutions. In such a setting, a key role in the arguments developed in [4] is played by the *compact embedding* of a related function space with radial symmetry into a certain class of Lebesgue spaces.

The approach developed in this paper is general and cannot be reduced to radially symmetric solutions, due to the presence of the general potential a. A central role in the arguments developed in [4] is played by the fact that the space $\mathcal{W}$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$, provided that $1 < p < \min\{q, N\}$, $1 < p^*q'/p'$ and $\alpha \in (1, p^*q'/p')$. By interpolation, the same continuous embedding holds in every Lebesgue space $L^r(\mathbb{R}^N)$ for every $r \in [\alpha, p^*]$.

3 Proof of the main result

We give the proof of Theorem 2.3 by using the following version of the mountain pass lemma of A. Ambrosetti and P. Rabinowitz [2] (see also H. Brezis and L. Nirenberg [9]).

Theorem 3.1. *Let $\mathcal{X}$ be a real Banach space and assume that $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ is a C^1-functional that satisfies the following geometric hypotheses:*

(i) $\mathcal{E}(0) = 0$ *and there exist positive numbers a and r such that $\mathcal{E}(u) \geq a$ for all $u \in \mathcal{X}$ with $\|u\| = r$;*

(ii) there exists $e \in \mathcal{X}$ with $\|e\| > r$ such that $\mathcal{E}(e) < 0$.

Set

$$\mathcal{P} := \{p \in C([0,1]; \mathcal{X});\ p(0) = 0,\ p(1) = e\}$$

and

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in [0,1]} \mathcal{E}(p(t)).$$

Then there exists a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n \to \infty} \mathcal{E}(u_n) = c \quad \text{and} \quad \lim_{n \to \infty} \|\mathcal{E}'(u_n)\|_{\mathcal{X}^*} = 0.$$

Moreover, if $\mathcal{E}$ satisfies the Palais–Smale condition at the level c, then c is a critical value of $\mathcal{E}$.

We split the proof into several steps.

3.1 Existence of a mountain and of a valley

Fix $r \in (0, 1)$ and let $u \in \mathcal{X}$ with $\|u\| = r$. Using hypotheses (ϕ_1) and (ϕ_2) we have

$$\mathcal{E}(u) \geq \frac{c_1}{2} \int_{[|\nabla u| \leq 1]} |\nabla u|^q dx + \frac{c_2}{2} \int_{[|\nabla u| > 1]} |\nabla u|^p dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} a(x) |u|^\alpha dx - \int_{\mathbb{R}^N} F(x, u) dx. \quad (3.1)$$

Fix $\varepsilon > 0$. Using hypotheses (f_1) and (f_2), there exists $C_\varepsilon > 0$ such that

$$|F(x,u)| \leq \varepsilon |u|^\alpha + C_\varepsilon |u|^{p^*} \quad \text{for all } u \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.2}$$

Setting $c := \min\{c_1/2, c_2/2\}$, relation (3.2) and hypothesis (a_1) yield

$$\begin{aligned} \mathcal{E}(u) &\geq c \max\left\{\int_{[|\nabla u| \leq 1]} |\nabla u|^q dx, \int_{[|\nabla u|>1]} |\nabla u|^p dx\right\} + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx \\ &\quad - \varepsilon \int_{\mathbb{R}^N} |u|^\alpha dx - C_\varepsilon \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\geq c\,\|\nabla u\|_{p,q}^q + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - \frac{\varepsilon}{a_0}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - C_\varepsilon \int_{\mathbb{R}^N} |u|^{p^*} dx. \end{aligned} \tag{3.3}$$

We have already observed that $\mathcal{X}$ is continuously embedded into the function space $\mathcal{W}$ defined in [3, p. 584]. Using now [3, Corollary 2.4], it follows that $\mathcal{X}$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$, hence there exists $C > 0$ such that

$$\|u\|_{p^*} \leq C\,\|u\| \quad \text{for all } u \in \mathcal{X}.$$

Returning to (3.3), we deduce that

$$\mathcal{E}(u) \geq c\,\|\nabla u\|_{p,q}^q + \left(\frac{1}{\alpha} - \frac{\varepsilon}{a_0}\right)\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - C\,\|u\|_{p^*}^{p^*}. \tag{3.4}$$

Recall that $\max\{\alpha, q\} < p^*$, see hypothesis (2.2). Taking $r \in (0,1)$ small enough, relation (3.4) yields that there is $a > 0$ such that

$$\mathcal{E}(u) \geq a \quad \text{for all } u \in \mathcal{X} \text{ with } \|u\| = r. \tag{3.5}$$

This shows the existence of a "mountain" around the origin.

Next, we prove the existence of a "valley" over the chain of mountains. This is essentially due to the relationship between the exponents p, q and α. For this purpose, we fix $w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Using hypothesis (ϕ_3) we have

$$\begin{aligned} \mathcal{E}(tw) &\leq \frac{c_2}{2}\int_{[|\nabla(tw)| \leq 1]} |\nabla(tw)|^q dx + \frac{c_2}{2}\int_{[|\nabla(tw)|>1]} |\nabla(tw)|^p dx \\ &\quad + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|tw|^\alpha dx - \int_{\mathbb{R}^N} F(x,tw)dx \\ &\leq \frac{c_2}{2}\left(t^q \int_{\mathbb{R}^N} |\nabla w|^q dx + t^p \int_{\mathbb{R}^N} |\nabla w|^p dx\right) + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N} a(x)|w|^\alpha dx \\ &\quad - \varepsilon t^\alpha \int_{\mathbb{R}^N} |w|^\alpha dx - C_\varepsilon t^{p^*}\int_{\mathbb{R}^N} |w|^{p^*} dx. \end{aligned} \tag{3.6}$$

Since w is fixed, relation (3.6) and hypothesis (2.2) imply that $\lim_{t\to+\infty} \mathcal{E}(tw) = -\infty$. Thus, there exists $t_0 > 0$ such that $\mathcal{E}(t_0 w) < 0$.

We have checked until now the geometric hypotheses of the mountain pass lemma. We argue in what follows that the corresponding setting is non-degenerate, that is, the associated min-max value given by Theorem 3.1 is positive.

Set

$$c := \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} \mathcal{E}(p(t)),$$

where

$$\mathcal{P} := \{p \in C([0,1];\mathcal{X});\ p(0) = 0,\ p(1) = t_0 w\}.$$

We observe that for all $p \in \mathcal{P}$

$$c \geq \mathcal{E}(p(0)) = \mathcal{E}(0) = 0.$$

In fact, we claim that

$$c > 0. \tag{3.7}$$

Arguing by contradiction, we assume that $c = 0$. In particular, this means that for all $\varepsilon > 0$ there exists $q \in \mathcal{P}$ such that

$$0 \leq \max_{t\in[0,1]} \mathcal{E}(q(t)) < \varepsilon\,.$$

Fix $\varepsilon < a$, where a is given by (3.5). Then $q(0) = 0$ and $q(1) = t_0 w$, hence

$$\|q(0)\| = 0 \quad \text{and} \quad \|q(1)\| > r.$$

Using the continuity of q, there exists $t_1 \in (0,1)$ such that $\|q(t_1)\| = r$, hence

$$\|\mathcal{E}(q(t_1))\| = a > \varepsilon,$$

which is a contradiction. This shows that our claim (3.7) is true.

Applying Theorem 3.1, we find a Palais–Smale sequence for the level $c > 0$, that is, a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n\to\infty} \mathcal{E}(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{E}'(u_n)\|_{\mathcal{X}^*} = 0. \tag{3.8}$$

3.2 The boundedness of the Palais–Smale sequence

We prove in what follows that the sequence (u_n) described in (3.8) is bounded in $\mathcal{X}$. Indeed, using both information in relation (3.8), we have

$$\begin{aligned} c + O(1) + o(\|u_n\|) &= \mathcal{E}(u_n) - \frac{1}{\theta}\mathcal{E}'(u_n)u_n \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{\theta}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\ &\quad + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n)\right] dx. \end{aligned} \tag{3.9}$$

Using hypothesis (f_3), relation (3.9) yields

$$\begin{aligned} c + O(1) + o(\|u_n\|) &= \mathcal{E}(u_n) - \frac{1}{\theta}\mathcal{E}'(u_n)u_n \\ &\geq \int_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{\theta}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\ &\quad + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx. \end{aligned} \tag{3.10}$$

Using hypothesis (ϕ_4) we have for all $t \geq 0$

$$\frac{1}{2}\phi(t) - \frac{1}{\theta}\phi'(t)t \geq \frac{1-\mu s}{2}\phi(t),$$

where $\mu s \in (0,1)$. Hypothesis (f_3) yields that $\theta > \alpha$. Thus, returning to (3.10) we deduce that there exists $c_0 > 0$ such that for all $n \in \mathbb{N}$

$$\mathcal{E}(u_n) - \frac{1}{\theta}\mathcal{E}'(u_n)u_n \geq c_0 \left[\min\{\|\nabla u_n\|^q_{p,q}, \|\nabla u_n\|^p_{p,q}\} + \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx\right]. \tag{3.11}$$

Combining relations (3.10) and (3.11), we deduce that the sequence $(u_n) \subset \mathcal{X}$ is bounded.

Since $\mathcal{X}$ is a closed subset of $\mathcal{W}$, using Proposition 2.5 in [4] we deduce that the sequence (u_n) converges weakly (up to a subsequence) in $\mathcal{X}$ and strongly in $L^s_{\rm loc}(\mathbb{R}^N)$ to some u_0. We show in what follows that u_0 is a solution of problem (2.1).

Fix $\zeta \in C^\infty_c(\mathbb{R}^N)$ and set $\Omega := \operatorname{supp}(\zeta)$. Define

$$A(u) = \frac{1}{2}\int_\Omega \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_\Omega a(x)|u|^\alpha dx$$

and

$$B(u) = \int_\Omega F(x,u)dx.$$

Using (3.8) we have

$$A'(u_n)(\zeta) - B'(u_n)(\zeta) \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Since $u_n \to u_0$ in $L^s(\Omega)$ and the mapping $u \mapsto F(x,u)$ is compact from $\mathcal{X}$ into L^1, it follows that

$$B(u_n) \to B(u_0) \quad \text{and} \quad B'(u_n)(\zeta) \to B'(u_0)(\zeta) \quad \text{as } n \to \infty. \tag{3.13}$$

Combining relations (3.12) and (3.13) we deduce that

$$A'(u_n)(\zeta) \to B'(u_0)(\zeta) \quad \text{as } n \to \infty. \tag{3.14}$$

Using hypothesis (ϕ_5), we obtain that the nonlinear mapping A is convex. Therefore

$$A(u_n) \leq A(u_0) + A'(u_n)(u_n - u_0) \quad \text{for all } n \in \mathbb{N}. \tag{3.15}$$

Using (3.14) in combination with $u_n \rightharpoonup u_0$ in $\mathcal{X}$, relation (3.15) yields

$$\limsup_{n\to\infty} A(u_n) \leq A(u_0).$$

But A is lower semicontinuous, since it is convex and continuous. It follows that

$$A(u_0) \leq \liminf_{n\to\infty} A(u_n).$$

We conclude that

$$A(u_n) \to A(u_0) \quad \text{as } n \to \infty.$$

From now on, with the same arguments as in [4, p. 210] (see also [12, p. 59]), we deduce that

$$\nabla u_n \to \nabla u_0 \quad \text{as } n \to \infty \text{ in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \to \int_{\mathbb{R}^N} a(x)|u_0|^\alpha dx \quad \text{as } n \to \infty.$$

Therefore

$$\int_\Omega \phi'(|\nabla u_0|^2)\nabla u_0 \nabla\zeta dx + \int_\Omega a(x)|u_0|^{\alpha-2}u_0\zeta dx - \int_\Omega f(x,u_0)\zeta dx = 0.$$

By density, we obtain that this identity holds for all $\zeta \in \mathcal{X}$, hence u_0 is a solution of problem (2.1).

3.3 Proof of Theorem 2.3 completed

It remains to argue that the solution u_0 is nontrivial. For this purpose we use some ideas developed in [12,14].

Using the fact that (u_n) is a Palais–Smale sequence, relation (3.8) implies that if n is a positive integer sufficiently large then

$$\begin{aligned}\frac{c}{2} &\leq E(u_n) - \frac{1}{2}E'(u_n)u_n \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left[\phi(|\nabla u_n|^2) - \phi'(|\nabla u_n|^2)|\nabla u_n|^2\right]dx \\ &\quad + \left(\frac{1}{\alpha}-\frac{1}{2}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx.\end{aligned} \tag{3.16}$$

Using hypothesis (ϕ_5) that concerns the convexity of the map $t \mapsto \phi(t^2)$, we deduce that

$$\phi(t^2) - \phi(0) \leq \phi'(t^2)t^2.$$

Using now (ϕ_1) we obtain

$$\phi(t^2) \leq \phi'(t^2)t^2,$$

hence

$$\phi(|\nabla u_n|^2) \leq \phi'(|\nabla u_n|^2)|\nabla u_n|^2. \tag{3.17}$$

We first assume that $\alpha \geq 2$. Thus, relations (3.16) and (3.17) combined with hypothesis (f_3) imply that for all n large enough we have

$$\begin{aligned}\frac{c}{2} &\leq \int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n)u_n dx.\end{aligned} \tag{3.18}$$

Hypotheses (f_1) and (f_2) show that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u|^{p^*-1} + C_\varepsilon |u|^{\alpha-1}, \quad \text{for all } u \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^N.$$

Returning to (3.18) we obtain for all n large enough

$$\frac{c}{2} \leq \frac{\varepsilon}{2}\|u_n\|_{p^*}^{p^*} + C_\varepsilon \|u_n\|_\alpha^\alpha .$$

Since (u_n) is bounded in $L^{p^*}(\mathbb{R}^N)$, we fix $\varepsilon > 0$ small enough such that

$$\frac{\varepsilon}{2}\sup_n \|u_n\|_{p^*}^{p^*} \leq \frac{c}{4}.$$

It follows that for all $n \geq n_0$

$$\frac{c}{4} \leq C_0 \|u_n\|_\alpha^\alpha ,$$

where C_0 is a positive constant.

In order to show that $u_0 \neq 0$ we argue by contradiction. Assume that $u_0 = 0$. In particular, this implies that

$$u_n \to 0 \quad \text{in } L^\alpha_{\mathrm{loc}}(\mathbb{R}^N). \tag{3.19}$$

Let k be a positive integer and set

$$\omega := \{x \in \mathbb{R}^N;\ 1/k < |x| < k\}. \tag{3.20}$$

Using (3.19), it follows that if k is large enough then

$$C_0 \int_\omega |u_n|^\alpha dx \le \frac{c}{8} \quad \text{for all } n \ge n_0\,.$$

Therefore

$$\begin{aligned}
\frac{c}{8} &\le C_0 \int_{\mathbb{R}^N\setminus\omega} |u_n|^\alpha dx \\
&\le \frac{C_0}{\inf_{|x|\le 1/k} a(x)} \int_{|x|\le 1/k} a(x)|u_n|^\alpha dx + \frac{C_0}{\inf_{|x|\ge k} a(x)} \int_{|x|\ge k} a(x)|u_n|^\alpha dx \\
&\le C_0 M \left[\frac{1}{\inf_{|x|\le 1/k} a(x)} + \frac{1}{\inf_{|x|\ge k} a(x)}\right],
\end{aligned} \tag{3.21}$$

where $M = \sup_n \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx$.

Choosing k large enough and using hypothesis (a_2), relation (3.21) implies that $c = 0$, a contradiction.

It remains to study the case $1 < \alpha < 2$. Relations (3.16) and (3.17) imply that for all n large enough we have

$$\begin{aligned}
\frac{c}{2} &\le E(u_n) - \frac{1}{2}E'(u_n)u_n \\
&\le \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n)u_n dx.
\end{aligned} \tag{3.22}$$

We argue again by contradiction and assume that $u_0 = 0$. With the same choice of ω as in (3.20) and with similar estimates in (3.22) as above, we obtain a contradiction.

Summarizing, we have obtained that u_0 is a nontrivial solution of problem (2.1). □

3.4 Final remarks

(i) The study of Orlicz spaces $L^p + L^q$ has been initiated by M. Badiale, L. Pisani, and S. Rolando [6].

(ii) We point out that with a similar analysis we can treat the case of potentials satisfying $\liminf_{|x|\to\infty} a(x) = 0$, which is a particular *critical frequency case*, see for details J. Byeon and Z. Q. Wang [8].

(iii) The existence of solutions of problem (2.1) in the case of a null potential a was established by H. Berestycki and P. L. Lions [7], where the authors used a *double-power* growth hypothesis on the nonlinearity, that is, $f(x,\cdot)$ has a subcritical behaviour at infinity and a supercritical growth near the origin.

(iv) We expect that new and interesting results can be established if the nonhomogeneous operator in problem (2.1) is a replaced by a differential operator with *two* competing potentials ϕ_1 and ϕ_2. We refer to operators of the type

$$\operatorname{div}\left((\phi_1'(|\nabla u|^2) + \phi_2'(|\nabla u|^2))|\nabla u|^2\right),$$

where ϕ_1 and ϕ_2 have different growth decay. This new abstract framework is inspired by the analysis developed in Chapter 3.3 of the recent monograph by V. Rădulescu and D. Repovš [23] in the framework of nonlinear problems with *variable exponents*.

(v) A new research direction in strong relationship with several relevant applications is the study of problems described by the nonlocal term

$$M\left[\int_{\mathbb{R}^N} \phi(|\nabla u|^2)|\nabla u|^2\right].$$

We refer here to the pioneering papers by P. Pucci *et al.* [5,20,21] related to Kirchhoff problems involving nonlocal operators associated to the standard Laplace, p-Laplace or $p(x)$-Laplace operators.

Acknowledgements

The first author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No (RG-1435-026). The second author thanks the Visiting Professor Programming at King Saud University for funding this work.

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